Singular kinetic equations

Xiangchan Zhu
(A joint work with Zimo Hao, Xicheng Zhang and Rongchan Zhu)
Chinese Academy of Science

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Motivation-(Mean field limit)

- Consider the following N-particle systems:

$$\begin{cases} \mathrm{d}X_t^i = V_t^i \mathrm{d}t, \\ \mathrm{d}V_t^i = b(Z_t^i) + \frac{1}{N}\sum_{j\neq i} K(X_t^i - X_t^j) + \sqrt{2}\mathrm{d}B_t^i, \end{cases}$$

where $i = 1, 2, .., N$,
$Z^i = (X^i, V^i) \in \mathbb{R}^{2d}$: position and velocity of particle number i
B_t^i: independent Brownian motions
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$$\partial_t u = \Delta_v u + v \cdot \nabla_x u + b \cdot \nabla_v u + K * \langle u \rangle \cdot \nabla_v u, \quad u(0) = u_0, \tag{1}$$

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- Problem: For b singular, (e.g. spatial white noise), global well-posedness of (1)?

DDSDE

- When b, K are smooth, the solution of the Fokker-Planck equation (1) is the density of the following Distribution Dependent SDE(DDSDE):

$$\begin{cases} \mathrm{d}X_t = V_t \mathrm{d}t \\ \mathrm{d}V_t = b(Z_t)\mathrm{d}t + \int_{\mathbb{R}^d} K(X_t - y)\mu_t(\mathrm{d}y)\mathrm{d}t + \mathrm{d}B_t \\ Z_0 \sim u_0 \mathrm{d}x\mathrm{d}v, \end{cases} \tag{2}$$

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- Problem: For b singular, (e.g. spatial white noise), global well-posedness of (2)? Nonlinear martingale problem.

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- Consider the following equation

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  $\Rightarrow$ the best regularity of the solution is in $L_T^\infty \mathbf{C}_a^{2-\alpha}$ with $\mathbf{C}_a^{2-\alpha}$ anisotropic Besov space.

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$$\mathbf{C}_a^{\alpha} \times \mathbf{C}_a^{\beta} \ni (f, g) \to fg \in \mathbf{C}_a^{\alpha \wedge \beta} \text{ only if } \alpha + \beta > 0.$$

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  Hairer 14 the theory of regularity structures
  Gubinelli, Imkeller and Perkowski 15 : paracontrolled distribution method
- Aim: develop paracontrolled calculus to get global well-posedness of (2)

Linear equation

Linear equation

- For $\lambda \geqslant 0$, we consider the following linear PDE:

$$\mathscr{L}_\lambda u := (\partial_t - \Delta_v - v \cdot \nabla_x + \lambda)u = b \cdot \nabla_v u + f, \quad u(0) = u_0. \tag{3}$$

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- Second difficulty: Loss of weight from $b \cdot \nabla_v u$
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- Aim: develop paracontrolled distribution method in the kinetic setting to obtain Schauder estimate for (3).

Kinetic Hölder space and Schauder estimate

Define

$$\Gamma_t f(z) := f(\Gamma_t z), \quad \Gamma_t z := (x + tv, v).$$

Let $\alpha \in (0,2)$ and $T > 0$. Define

$$\mathbb{S}^{\alpha}_{T,a}(\rho) := \Big\{ f : \|f\|_{\mathbb{S}^{\alpha}_{T,a}(\rho)} := \|f\|_{L^\infty_T \mathbf{C}^{\alpha}_a(\rho)} + \|f\|_{\mathbf{C}^{\alpha/2}_{T;\Gamma} L^\infty(\rho)} < \infty \Big\},$$

where for $\beta \in (0,1)$,

$$\|f\|_{\mathbf{C}^{\beta}_{T;\Gamma} L^\infty(\rho)} := \sup_{0 \leqslant t \leqslant T} \|f(t)\|_{L^\infty(\rho)} + \sup_{0 < |t-s| \leqslant 1} \frac{\|f(t) - \Gamma_{t-s} f(s)\|_{L^\infty(\rho)}}{|t-s|^{\beta}}.$$

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Let $\mathscr{I}_\lambda = (\mathscr{L}_\lambda)^{-1}$:

Lemma 2.1

[Schauder estimates] Let $\beta \in (0,2)$ and $\theta \in (\beta, 2]$. For any $q \in [\frac{2}{2-\theta}, \infty]$ and $T > 0$, there is a constant $C = C(d, \beta, \theta, q, T) > 0$ such that for all $\lambda \geq 0$ and $f \in L^q_T \mathbf{C}^{-\beta}_a(\rho)$,

$$\|\mathscr{I}_\lambda f\|_{\mathbb{S}^{\theta-\beta}_{T,a}(\rho)} \lesssim_C (\lambda \vee 1)^{\frac{\theta}{2} + \frac{1}{q} - 1} \|f\|_{L^q_T \mathbf{C}^{-\beta}_a(\rho)}.$$

Paraproducts

- Bony's decomposition: $f=\sum_{i\geqslant -1}\Delta_i f$, $\widehat{\Delta_i f}=\phi_i^a\hat{f}$, $\{\phi_i^a\}_{i\geqslant -1}$,

$$fg=\sum_{i\geqslant -1}\Delta_i f\sum_{j\geqslant 0}\Delta_j g=f\prec g+f\circ g+f\succ g,$$

where

$$f\prec g=g\succ f:=\sum_{j\geqslant 0}\sum_{i<j-1}\Delta_i f\Delta_j g$$

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- $f \prec g$ always well defined but regularity not better than g.
- $f \succ g$, $f \circ g$ regularity become better if f is regular.

Paracontrolled solution to linear PDE

- $$\mathscr{L}_\lambda u = b \cdot \nabla_v u + f = \underbrace{\nabla_v u \prec b}_{\text{bad term}} + \nabla u \succ b + \underbrace{b \circ \nabla_v u}_{\text{not well defined}} + f$$

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- Paracontrolled solution:

$$u = \nabla_v u \prec \mathscr{I}_\lambda b + \underbrace{u^\sharp}_{\text{regular term}} + \mathscr{I}_\lambda f, \quad \text{paracontrolled ansatz}$$

$$u^\sharp = \mathscr{I}_\lambda(\nabla_v u \succ b + b \circ \nabla_v u) - [\mathscr{I}_\lambda, \nabla_v u \prec]b.$$

Commutator estimate for kinetic operator

Let P_t be the kinetic semigroup.

Lemma 2.2

For any $\alpha\in(0,1)$, $\beta\in\mathbb{R}$, $t\in(0,T]$, $\delta\geqslant 0$, $j\geqslant -1$,

$$\|\Delta_j[P_t(f\prec g)-(\Gamma_t f\prec P_t g)]\|_{L^\infty(\rho_1\rho_2)}\lesssim t^{-\frac{\delta}{2}}2^{-(\alpha+\beta+\delta)j}\|f\|_{\mathbf{C}_a^\alpha(\rho_1)}\|g\|_{\mathbf{C}_a^\beta(\rho_2)}.$$

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$\Rightarrow$

Lemma 2.3

Commutator estimate

$$\|[\mathscr{I}_\lambda, f \prec]g\|_{L^\infty_T \mathbf{C}_a^{\alpha+\beta+2}(\rho_1\rho_2)} \lesssim \| f\|_{\mathbb{S}^\alpha_{T,a}(\rho_1)} \|g\|_{L^\infty_T \mathbf{C}^\beta_a(\rho_2)}. \tag{4}$$

$\Rightarrow u \in C_T \mathbf{C}_a^{2-\alpha}(\rho_\delta), u^\sharp \in C_T \mathbf{C}_a^{3-2\alpha}(\rho_\delta)$

Renormalization and well-posedness of linear PDE

- If $b \circ \nabla_v \mathscr{I}_\lambda b, b \circ \nabla_v \mathscr{I}_\lambda f \in L^\infty_T \mathbf{C}^{1-2\alpha}_a(\rho_\kappa)$

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- Let b be a Gaussian field with the following covariance:

$$\mathbb{E}\big(b(g_1)b(g_2)\big)=\int_{\mathbb{R}^{2d}}\hat{g}_1(\zeta)\,\hat{g}_2(-\zeta)\mu(\mathrm{d}\zeta).$$

Assumption: μ is symmetric in second variable and for some $\beta\in(\frac{1}{2},\frac{2}{3})$,

$$\sup_{\zeta'\in\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}\frac{\mu(\mathrm{d}\zeta)}{(1+|\zeta'+\zeta|_a)^{2\beta}}<\infty.$$

Probabilistic calculation $\Rightarrow b\circ\nabla_v\mathscr{I}_\lambda b\in L^\infty_T\mathbf{C}^{1-2\alpha}_a(\rho_\kappa)$

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Interesting point: 0th Wiener chaos is not zero but there's no renormalization term

Theorem 1

Let $\alpha\in(\frac{1}{2},\frac{2}{3})$ and $\vartheta:=\frac{9}{2-3\alpha}$ and $\delta:=(2\vartheta+2)\kappa\leqslant 1$. For any $T>0$, (b,f) as above, $\exists!$ paracontrolled solution $(u,u^\sharp)$ to PDE (3) such that $\|u\|_{C_T\mathbf{C}^{2-\alpha}_T(\rho_\delta)}+\|u^\sharp\|_{C_T\mathbf{C}^{3-2\alpha}_T(\rho_{2\delta})}\lesssim C(b,f)$.

Nonlinear equation

Nonlinear mean field equation

- Consider the following

$$\mathscr{L}u = b \cdot \nabla_v u + K * \langle u \rangle \cdot \nabla_v u, \quad u(0) = u_0.$$

  Here $\langle u \rangle(t,x) := \int_{\mathbb{R}^d} u(t,x,v)\mathrm{d}v$. Assume that

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$$K \in \cup_{\beta>\alpha-1} \mathbf{C}_x^{\beta/3}, b \circ \nabla_v \mathscr{I}(b) \in \mathbf{C}_a^{1-2\alpha}(\rho_\kappa)$$

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Theorem 2

Let $\alpha \in (\frac{1}{2}, \frac{2}{3})$ and κ be small enough so that $\delta := 2(\frac{9}{2-3\alpha} + 1)\kappa < 1$.

- *for any probability density $u_0 \in L^1(\rho_0) \cap \mathbf{C}_a^\gamma$, $\gamma > 1 + \alpha$, $\exists$ at least a probability density paracontrolled solution $u \in L_T^\infty(\mathbf{C}^{2-\alpha}(\rho_\delta))$ to nonlinear mean field equation.*
- *If in addition that K is bounded, then for any initial data $u_0 \in L^1(\rho_0) \cap \mathbf{C}_a^\gamma$ with $e^{-\rho_0} \in L^1$ satisfying $H(u_0) := \int u_0 \ln u_0 < \infty$, the solution is unique.*

Idea of proof

- A priori estimate: Linear approximation and use Theorem 1

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- Existence: approximation by convolution with smooth modifier
- Uniqueness: Linear approximation and a priori estimate of $\|\nabla_v u\|^2_{L^2_t L^1}$ and L^1 estimate

Singular DDSDE

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- Consider the following kinetic DDSDE with singular drift:

$$\mathrm{d}X_t = V_t \mathrm{d}t, \;\; \mathrm{d}V_t = b(X_t, V_t)\mathrm{d}t + (K * \mu_{X_t})(X_t)\mathrm{d}t + \sqrt{2}\mathrm{d}B_t, \tag{5}$$

- B_t: a d-dimensional Brownian motion
- μ_{X_t}: law of X_t
- $K * \mu(x) := \int_{\mathbb{R}^d} K(x-y)\mu(\mathrm{d}y)$.
- b is singular

Problem: How to understand (5)?
Consider the following linear equation for given $\mu : [0, T] \to \mathcal{P}(\mathbb{R}^{2d})$

$$(\partial_t + \Delta_v + v \cdot \nabla_x)u + b \cdot \nabla_v u + K * \mu_t \cdot \nabla_v u = f, \;\; u(T) = \varphi. \tag{6}$$

Definition 4.1

(Martingale problem) Let $\delta > 0$. A probability measure $\mathbb{P} \in \mathcal{P}(\mathcal{C}_T)$ is called a martingale solution to SDE (5) *starting from $\nu \in \mathcal{P}_\delta(\mathbb{R}^{2d})$, if $\mathbb{P} \circ Z_0^{-1} = \nu$ and for all $f \in C_b([0, T] \times \mathbb{R}^{2d})$, $\varphi \in \mathbf{C}_a^\gamma(\mathbb{R}^{2d})$ with some $\gamma > 1 + \alpha$ and $\mu_t := \mathbb{P} \circ X_t^{-1}$,*

$$M_t := u_f^\mu(t, Z_t) - u_f^\mu(0, Z_0) - \int_0^t f(s, Z_s)\mathrm{d}s$$

is a martingale under $\mathbb{P}$ with respect to $(\mathscr{B}_t)$. Here u_f^μ is a solution to (6).

Main results

Theorem 3

Suppose that $b \circ \nabla_v \mathscr{I}(b) \in \mathbf{C}_a^{1-2\alpha}(\rho_\kappa)$ *and* $K \in \cup_{\beta>\alpha-1} \mathbf{C}_a^{\beta}$. *Then there exists at least one martingale solution* $\mathbb{P}$ *to SDE* (5). *Moreover, if* K *is bounded measurable, then the solution is unique.*

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Idea of proof

- Existence: approximation by convolution with smooth modifier
- Uniqueness: First for $K = 0$ and Girsanov's tansformation

Thank you !